3 Sobolev Spaces

Exercise 3.1. Since the function $\mathbb{R} \ni v \mapsto |v|^p$ is convex, by Jensen's inequality we have that for every $x \in Q_i$ it holds

$$|u_\delta(x)|^p = \left| \frac{1}{\delta^d} \int_{Q_i} u \right|^p \leq \frac{1}{\delta^d} \int_{Q_i} |u|^p.$$

Therefore

$$\|u_\delta\|_{L^p}^p = \sum_{i=1}^{\infty} \int_{Q_i} |u_\delta(x)|^p dx \leq \sum_{i=1}^{\infty} \int_{Q_i} |u|^p = \|u\|_{L^p}^p.$$

Let $\varepsilon > 0$ and $v \in C_c^\infty(\mathbb{R}^d)$ be such that $\|u - v\|_{L^p} < \varepsilon$. Then

$$\begin{aligned} \|u - u_\delta\|_{L^p} \leq & \|u - v\|_{L^p} + \|v - v_\delta\|_{L^p} + \|v_\delta - u_\delta\|_{L^p} \\ \leq & \varepsilon + \|v - v_\delta\|_{L^p} + \|(v - u)_\delta\|_{L^p} \\ \leq & \varepsilon + \|v - v_\delta\|_{L^p} + \|v - u\|_{L^p} \\ \leq & 2\varepsilon + \|v - v_\delta\|_{L^p}. \end{aligned} \tag{1}$$

Since v is Lipschitz we have that $\|v - v_\delta\|_{L^\infty} \leq n\mathrm{Lip}(v)\delta$ and since v has compact support, then $\|v - v_\delta\|_{L^\infty} \to 0$ implies that $\|v - v_\delta\|_{L^p} \to 0$ for every $p \in [1, \infty)$. In particular, by (1), it holds

$$\limsup_{\delta \to 0} \|u - u_\delta\|_{L^p} \leq 2\varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this shows that $u_\delta \to u$ as $\delta \to 0$.

Exercise 3.2. We first assume that $\mathfrak{F}$ is relatively compact in $L^p(\Omega)$. Assume by contradiction that there exist $\varepsilon > 0$ and a sequence $(u_n) \subset \mathfrak{F}$ such that for every $n \in \mathbb{N}$ it holds $\|u_n - (u_n)_{1/n}\|_{L^p(\Omega)} \geq \varepsilon$. By compactness there exists a subsequence (that we do not relabel) $u_k \to \overline{u}$ in $L^p(\Omega)$ for some $\overline{u} \in L^p(\Omega)$. In particular there exists $\overline{k} \in \mathbb{N}$ such that for every $k > \overline{k}$ it holds $\|u_k - \overline{u}\|_{L^p(\Omega} < \varepsilon/3$. By the previous exercise we have that there exists $\widetilde{k}$ such that for every $k > \widetilde{k}$ it holds $\|\overline{u} - \overline{u}_{1/k}\|_{L^p(\Omega)} < \varepsilon/3$, therefore for $k > \overline{k}, \widetilde{k}$, we get the contradiction

$$\varepsilon < \|u_k - (u_k)_{1/k}\|_{L^p(\Omega)} \leq \|u_k - \overline{u}\|_{L^p(\Omega)} + \|\overline{u} - \overline{u}_{1/k}\|_{L^p(\Omega)} + \|\overline{u}_{1/k} - (u_k)_{1/k}\|_{L^p(\Omega)} < \varepsilon.$$

Let us now prove the converse implication : since $L^p(\Omega)$ is a complete metric space, relative compactness is equivalent to total boundedness. Therefore given $\varepsilon > 0$, we look for a finite number of balls of radius ε in $L^p(\Omega)$ which cover $\mathfrak{F}$. Let us first choose $\overline{\delta}$ relative to $\varepsilon/3$ and let us cover Ω with finitely many cubes $\{Q_i\}_{i=1}^N$ of side $\overline{\delta}$ as in the previous exercise.

Since $\mathfrak{F}$ is bounded, then there exists $K > 0$ (depending on $\overline{\delta}$) such that for every $u \in \mathfrak{F}$ it holds $\|u_{\overline{\delta}}\|_{L^\infty(\Omega)} < K$. We now propose a finite subset $F \subset L^p(\Omega)$ and we will check that $\mathfrak{F} \subset \bigcup_{v\in F} B_\varepsilon(v)$, where $B_\varepsilon(u)$ denotes the ball of center u and radius ε in $L^p(\Omega)$, namely that for every $u \in \mathfrak{F}$ there exists $v \in F$ such that $\|u - v\|_{L^p(\Omega)} < \varepsilon$. Let $\nu > 0$ be a constant that we will choose later, the candidate F is defined by

$$F = \{v \in L^p(\Omega) : v \text{ is constant in each } Q_i \text{ and } v(x) \in \nu\mathbb{Z} \cap [-K, K] \,\forall x \in \Omega\}.$$

Given $u \in \mathfrak{F}$, we can take $v \in F$ such that $\|v - u_{\overline{\delta}}\|_{L^\infty(\Omega)} < \nu$. Let us estimate

$$\|u - v\|_{L^p(\Omega)} \leq \|u - u_{\overline{\delta}}\|_{L^p(\Omega)} + |\Omega|^{\frac{1}{p}} \|u_{\overline{\delta}} - v\|_{L^\infty(\Omega)} \leq \frac{\varepsilon}{3} + |\Omega|^{\frac{1}{p}}\nu.$$

Therefore, taking $\nu = \frac{\varepsilon}{3|\Omega|^{\frac{1}{p}}}$, we have that $\|u - v\|_{L^p(\Omega)} < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this shows that $\mathfrak{F}$ is totally bounded and concludes the proof of this implication.

Let us now prove the second point : let Q be a cube of side $\delta > 0$ and assume first that $u \in W^{1,p}(Q) \cap C^\infty(Q)$. Then

$$\|u - u_\delta\|_{L^1(Q)} = \int_Q \left| u(x) - \frac{1}{\delta^d} \int_Q u(y)dy \right| dx \leq \int_Q \frac{1}{\delta^d} \int_Q |u(x) - u(y)|dydx. \tag{2}$$

We estimate $|u(x) - u(y)| \leq \int_0^1 |Du(x + t(y - x))||y - x|dt$ and since for every $x, y \in Q$, we have $|x - y| \leq \delta\sqrt{n}$, we can continue the estimate (2)

$$\begin{aligned}
\|u - u_\delta\|_{L^1(Q)} \leq & \frac{\sqrt{n}}{\delta^{n-1}} \int_Q \int_Q \int_0^1 |Du(x + t(y - x))||y - x|dtdydx \\
= & \frac{\sqrt{n}}{\delta^{n-1}} \int_0^1 \int_Q \int_Q |Du(x + t(y - x))||y - x|dydxdt \\
= & \frac{\sqrt{n}}{\delta^{n-1}} \int_0^1 \frac{1}{t^d} \int_Q \int_{(1-t)x+tQ} |Du(z)|dzdxdt \\
= & \frac{\sqrt{n}}{\delta^{n-1}} \int_0^1 \frac{1}{t^d} \int_Q \int_Q |Du(z)| \mathbf{1}_{\{(1-t)x+tQ\}}(z)dzdxdt,
\end{aligned} \tag{3}$$

where we used Fubini, the change of variable $z = x + t(y - x)$ and in the last line we observed that, since Q is convex, for every $x, y \in Q$ and every $t \in [0, 1]$, the point $x + t(y - x) \in Q$. We want to apply Fubini to exchange the integrals in dx and dz, so we observe that

$$z \in \{(1 - t)x + tQ\} \quad \Longleftrightarrow \quad x \in \frac{z}{1 - t} - \frac{t}{1 - t}Q.$$

We continue in (3) :

$$\begin{aligned}\|u-u_\delta\|_{L^1(Q)} \leq& \frac{\sqrt{n}}{\delta^{n-1}}\int_0^1 \frac{1}{t^d}\int_Q |Du(z)|\int_Q \mathbf{1}_{\frac{z}{1-t}-\frac{t}{1-t}Q}(x)dxdzdt\\ =&\frac{\sqrt{n}}{\delta^{n-1}}\int_0^1 \frac{1}{t^d}\int_Q |Du(z)|\mathcal{L}^d\left(Q\cap\frac{z}{1-t}-\frac{t}{1-t}Q\right)dzdt \qquad (4)\\ \leq&\frac{\sqrt{n}}{\delta^{n-1}}\int_0^1 \frac{1}{t^d}\min\left\{\mathcal{L}^d(Q),\mathcal{L}^d\left(\frac{t}{1-t}Q\right)\right\}\int_Q |Du(z)|dzdt.\end{aligned}$$

For $t\leq\frac{1}{2}$, we have $t/(1-t)\leq 2t\leq 1$ so that $\min\left\{\mathcal{L}^d(Q),\mathcal{L}^d\left(\frac{t}{1-t}Q\right)\right\}=(2t)^d\mathcal{L}^d(Q)$ and for $t>\frac{1}{2}$, we have $\min\left\{\mathcal{L}^d(Q),\mathcal{L}^d\left(\frac{t}{1-t}Q\right)\right\}=\mathcal{L}^d(Q)\leq(2t)^d\mathcal{L}^d(Q)$. We therefore continue in (5) :

$$\begin{aligned}\|u-u_\delta\|_{L^1(Q)} \leq& \frac{\sqrt{d}}{\delta^{d-1}}\int_0^1\frac{1}{t^d}(2t)^d\mathcal{L}^d(Q)\int_Q|Du(z)|dzdt\\ =&2^d\sqrt{d}\delta\|Du\|_{L^1(Q)}.\end{aligned} \qquad (5)$$

So far we showed the estimate for smooth functions inside a cube. In order to deal with the general case let $\Omega'\subset\mathbb{R}^d$ be a bounded set such that $\Omega+B(0,1)\subset\Omega'$ in such a way that there exists an extension operator $E:W^{1,p}(\Omega)\to W_0^{1,p}(\Omega')$. Being $C_c^\infty(\Omega')$ dense in $W_0^{1,p}(\Omega')$, we can find $(u_\varepsilon)_\varepsilon\subset C_c^\infty(\Omega')$ such that $\|u_\varepsilon-E(u)\|_{W^{1,p}(\Omega')}\to 0$. Moreover, for every $\lambda>0$, we can find a set Ω_λ such that

1. $\Omega\subset\Omega_\lambda\subset\subset\Omega'$;
2. Ω_λ is the union of essentially disjoint n-cubes of side σ, where $\sigma=\sigma(\lambda)$ is chosen so small that
   — $\sigma<\delta$;
   — $\mathcal{L}^d(\Omega_\lambda\setminus\Omega)<\sigma$;
   — $\|DE(u)\|_{L^1(\Omega_\lambda\setminus\Omega)}<\lambda$.

We can then compute

$$\begin{aligned}\|u-u_\delta\|_{L^1(\Omega)} \leq&\|E(u)-u_\delta\|_{L^1(\Omega_\lambda)}\\ \leq&\|E(u)-u_\varepsilon\|_{L^1(\Omega_\lambda)}+\|u_\delta-u_\varepsilon\|_{L^1(\Omega_\lambda)}\\ \leq&\|E(u)-u_\varepsilon\|_{L^1(\Omega_\lambda)}+\sum_{Q\in\Omega_\lambda}\|u_\delta-u_\varepsilon\|_{L^1(Q)}\\ \leq&\|E(u)-u_\varepsilon\|_{L^1(\Omega_\lambda)}+2^d\sqrt{d}\sigma\sum_{Q\in\Omega_\lambda}\|Du_\varepsilon\|_{L^1(Q)}\\ \leq&\|E(u)-u_\varepsilon\|_{L^1(\Omega_\lambda)}+2^d\sqrt{d}\delta\|Du_\varepsilon\|_{L^1(\Omega)}+2^d\sqrt{d}\delta\|Du_\varepsilon\|_{L^1(\Omega_\lambda\setminus\Omega)},\end{aligned}$$

where in the fourth inequality we used the estimate proved before. Passing to the limit as $\varepsilon\to 0$ we get

$$\begin{aligned}\|u-u_\delta\|_{L^1(\Omega)} \leq&2^d\sqrt{d}\delta\|Du\|_{L^1(\Omega)}+2^d\sqrt{d}\delta\|DE(u)\|_{L^1(\Omega_\lambda\setminus\Omega)}\\ \leq&2^d\sqrt{d}\delta\|Du\|_{L^1(\Omega)}+2^d\sqrt{d}\delta\lambda\end{aligned}$$

and passing to the limit as $\lambda \to 0$ we get the claim

$$\|u - u_\delta\|_{L^1(\Omega)} \leq 2^d \sqrt{d} \delta \|Du\|_{L^1(\Omega)}.$$

Exercise 3.3. We can assume that $\Omega = (0, l) \times \mathbb{R}^{n-1}$ for some $l > 0$ since, up to translation Ω is contained in a set of the form $(0, l) \times \mathbb{R}^{n-1}$ and we can extend all the elements of $W_0^{1,p}(\Omega)$ to elements of $W_0^{1,p}((0, l) \times \mathbb{R}^{n-1})$ by setting them equal to 0 in $(0, l) \times \mathbb{R}^{n-1} \setminus \Omega$. By density of $C_c^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$ it is sufficient to prove the statement for $u \in C_c^\infty(\Omega)$ as long as we obtain a constant C independent of u. Let us denote by $x = (z, y)$ with $z \in (0, l)$, $y \in \mathbb{R}^{n-1}$, the coordinates of a generic point $z \in \Omega$. For every $(z, y) \in \Omega$ it holds

$$u(z, y) = \int_0^z \frac{\partial u}{\partial x_1}(t, y) dt,$$

so that

$$|u(z, y)| \leq \int_0^l |Du|(t, y) dt = \|Du(\cdot, y)\|_{L^1(0,l)} \leq \|Du(\cdot, y)\|_{L^p(0,l)} l^{1/p'}.$$

Notice that the estimate depends only on y and not on z, therefore we have

$$\begin{aligned}
\|u\|_{L^p(\Omega)}^p = \int_\Omega |u(z, y)|^p dz dy &= \int_{\mathbb{R}^{n-1}} \left(\int_0^l |u(z, y)|^p dz \right) dy \\
&\leq \int_{\mathbb{R}^{n-1}} \left(\int_0^l \left(\|Du(\cdot, y)\|_{L^p(0,l)}^p l^{p/p'} \right) dz \right) dy \\
&= l^{p/p'+1} \int_{\mathbb{R}^{n-1}} \|Du(\cdot, y)\|_{L^p(0,l)}^p dy \\
&= l^{p/p'+1} \|Du\|_{L^p(\Omega)}^p.
\end{aligned}$$

This proves the first estimate, the second one is an immediate consequence.

Exercise 3.4. We follow the strategy of the proof of Poincaré-Wirtinger inequality from the lecture.

Suppose by contradiction that there exists a sequence $(u_h)_{h \in \mathbb{N}}$ such that $|\{u_h = 0\}| \geq \alpha$ and $\|u_h\|_{L^p(\Omega)} > h \|Du_h\|_{L^p(\Omega)}$, and consider the renormalized sequence

$$v_h = \frac{u_h}{\|u_h\|_{L^p(\Omega)}}.$$

We have therefore that $\|v_h\|_{W^{1,p}(\Omega)} = \|v_h\|_{L^p(\Omega)} + \|Dv_h\|_{L^p(\Omega)} \leq 1 + \frac{1}{h}$, so that $(v_h)_h$ is a bounded sequence in $W^{1,p}(\Omega)$. By Rellich-Kondrasov theorem, there exists $\overline{u} \in L^p(\Omega)$ and a subsequence (that we do not relabel) such that $v_h \to \overline{v}$ in $L^p(\Omega)$. Since $\|v_h\|_{L^p(\Omega)} = 1$ for every h, then $\|\overline{v}\|_{L^p(\Omega)} = 1$. Moreover, since $\|Dv_h\|_{L^p(\Omega)} \to 0$ as $h \to \infty$, then exactly as in the proof of Poincaré-Wirtinger inequality, we deduce that $D\overline{v} = 0$, and, since Ω is connected this implies that $\overline{v}$ is constant. Since $|\{v_h = 0\}| \geq \alpha$, and the L^1 convergence implies the pointwise a.e. convergence up to subsequences, then also $|\{\overline{v} = 0\}| \geq \alpha$. Being $\overline{v}$ constant, this shows that $\overline{v} = 0$ and this contradicts $\|\overline{v}\|_{L^p(\Omega)} = 1$.

Exercise 3.5. The fact that $\|\cdot\|$ is a norm follows immediately from the fact that $\|\cdot\|_{L^p(\Omega)}$ is a norm and the linearity of the divergence operator, therefore we need to check that $W^{1,p}_{\mathrm{div}}(\Omega)$ is complete. Let $u^k = (u_1^k, \ldots, u_n^k)$ be a Cauchy sequence in $W^{1,p}_{\mathrm{div}}(\Omega)$. In particular u^k is a Cauchy sequence in $(L^p(\Omega))^d$ and $\mathrm{div}u^k$ is a Cauchy sequence in $L^p(\Omega)$. Therefore, being $(L^p(\Omega))^n$ and $L^p(\Omega)$ complete, there exist $\overline{u} \in (L^p(\Omega))^n$ and $v \in L^p(\Omega)$ such that $u^k \to \overline{u}$ in $(L^p(\Omega))^n$ and $\mathrm{div}u^k \to v$ in $L^p(\Omega)$. In order to complete the proof we need to prove that $v = \mathrm{div}\overline{u}$. For any test function $\varphi \in C_c^\infty(\Omega)$ and every $k \in \mathbb{N}$ it holds

$$\int_\Omega u^k \cdot \nabla\varphi dx = \int_\Omega \mathrm{div}u^k \varphi dx$$

and letting $k \to \infty$ on both sides we obtain

$$\int_\Omega \overline{u} \cdot \nabla\varphi dx = \int_\Omega v\varphi dx,$$

namely $v = \mathrm{div}\overline{u}$.

Exercise 3.6. We check that $|\cdot|_{W^{\theta,p}}$ is a seminorm, namely that it is positively homogeneous (trivial) and it satisfies the triangular inequality, then we show that the space is complete.

Given $f \in W^{\theta,p}(\mathbb{R}^d)$, we define $\psi(f) \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$ by

$$\psi(f)(x,y) = \frac{|f(x) - f(y)|}{|x-y|^{\theta + \frac{d}{p}}}.$$

Given $f, g \in W^{\theta,p}(\mathbb{R}^d)$, we have that $\psi(f+g) \leq \psi(f) + \psi(g)$ pointwise a.e., therefore by the triangular inequality in $L^p(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$\|\psi(f+g)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} \leq \|\psi(f) + \psi(g)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} \leq \|\psi(f)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} + \|\psi(g)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)},$$

namely the triangular inequality holds true.

Let f_n be a Cauchy sequence in $W^{\theta,p}(\mathbb{R}^d)$. Then f_n and $\psi(f_n)$ are Cauchy sequences in $L^p(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d \times \mathbb{R}^d)$ respectively. In particular they converge to some $f \in L^p(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$. We still need to check that $g = \psi(f)$. This is true because, up to subsequences, we can assume that f_n and $\psi(f_n)$ converge a.e.